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A Globally Asymptotically Stable Polynomial Vector Field with no Polynomial Lyapunov Function

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***Abstract*— We give a simple, explicit example of a two-dimensional polynomial vector field that is globally asymptotically stable but does not admit a polynomial Lyapunov function.**

I. Introduction and Main Result

Given a particular class of differential equations, a question of fundamental importance in stability analysis is to determine a class of Lyapunov functions whose existence is necessary and sufficient for proving stability. Arguably, the class of polynomial differential equations are among the most widely encountered in engineering and sciences. For these systems, it is most common (and most natural) to search for Lyapunov functions that are polynomials themselves. This approach has become further prevalent over the past decade due to the fact that techniques from sum of squares optimization [1] have provided for algorithms that given a polynomial system can efficiently search for a polynomial Lyapunov function ([1], [2]).

The question therefore naturally arises as to whether the existence of polynomial Lyapunov functions is necessary for stability of polynomial systems. Since polynomials can approximate smooth functions with arbitrary accuracy on compact regions, one can expect the answer to this question to be positive if certain notions of stability such as exponential stability on compact sets are of interest [3]. On the other hand, to the best of our knowledge, the question of whether globally asymptotically stable (GAS) polynomial systems admit polynomial Lyapunov functions has been open. In fact, a recent reference in the controls literature ends with the following statement [4], [5]:

> "Still unresolved is the fundamental question of whether *globally* stable vector fields will also admit sum-of-squares Lyapunov functions."

Of course, the fundamental question referred to here is on existence of a polynomial Lyapunov function. If one were to exist, then we could simply square it to get another polynomial Lyapunov function that is a sum of squares. In this paper, we settle the question by giving a remarkably simple counterexample. In view of the fact that globally asymptotically stable linear systems always admit quadratic Lyapunov functions, it is quite interesting to observe that the following vector field that is arguably "the next simplest system" to consider does not admit a polynomial Lyapunov function of any degree.

Theorem 1.1: Consider the polynomial vector field

$$\begin{aligned} \dot{x} &= -x + xy \\ \dot{y} &= -y. \end{aligned} \tag{1}$$

The origin is a globally asymptotically stable equilibrium point, but the system does not admit a polynomial Lyapunov function.

Proof: Let us first show that the system is GAS. Consider the Lyapunov function

$$V(x,y) = \ln(1+x^2) + y^2,$$

which clearly vanishes at the origin, is strictly positive for all $(x,y) \neq (0,0)$, and is radially unbounded. The derivative of $V(x,y)$ along the trajectories of (1) is given by

$$\begin{aligned} \dot{V}(x,y) &= \tfrac{\partial V}{\partial x}\dot{x} + \tfrac{\partial V}{\partial y}\dot{y} \\ &= \tfrac{2x^2(y-1)}{1+x^2} - 2y^2 \\ &= -\tfrac{x^2+2y^2+x^2y^2+(x-xy)^2}{1+x^2}, \end{aligned}$$

which is obviously strictly negative for all $(x,y) \neq (0,0)$. In view of classical Lyapunov stability theorems (see e.g. [6, p. 124]), this shows that the origin is globally asymptotically stable.

Let us now prove that no positive definite polynomial Lyapunov function (of any degree) can decrease along the trajectories of system (1). The proof will be based on simply considering the value of a candidate Lyapunov function at two specific points. We will look at trajectories on the nonnegative orthant, with initial conditions on the line $(k, \alpha k)$ for some constant $\alpha > 0$, and then observe the location of the crossing of the trajectory with the horizontal line $y = \alpha$. We will argue that by taking k large enough, the trajectory will have to travel "too far east" (see Figure 1) and this will make it impossible for any polynomial Lyapunov function to decrease.

To do this formally, we start by noting that we can explicitly solve for the solution $(x(t), y(t))$ of the vector field in (1) starting from any initial condition $(x(0), y(0))$:

$$\begin{aligned} x(t) &= x(0)e^{[y(0)-y(0)e^{-t}-t]} \\ y(t) &= y(0)e^{-t}. \end{aligned} \tag{2}$$

Consider initial conditions

$$(x(0), y(0)) = (k, \alpha k)$$

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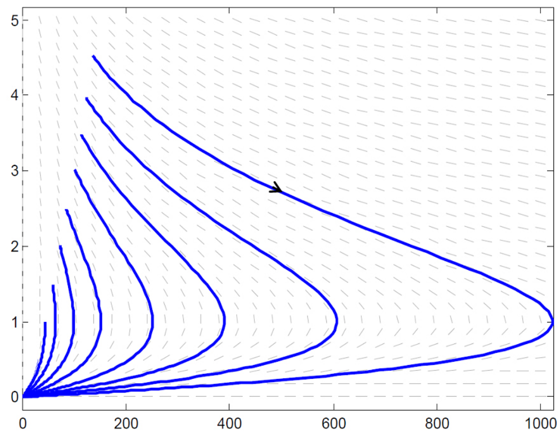

Fig. 1. Typical trajectories of the vector field in (1) starting from initial conditions in the nonnegative orthant.

parameterized by $k > 1$ and for some fixed constant $\alpha > 0$. From the explicit solution in (2) we have that the time t^* it takes for the trajectory to cross the line $y = \alpha$ is

$$t^* = \ln(k),$$

and that the location of this crossing is given by

$$(x(t^*), y(t^*)) = (e^{\alpha(k-1)}, \alpha).$$

Consider now any candidate nonnegative polynomial function $V(x, y)$ that depends on both x and y (as any Lyapunov function should). Since $k > 1$ (and thus, $t^* > 0$), for $V(x, y)$ to be a valid Lyapunov function, it must satisfy $V(x(t^*), y(t^*)) < V(x(0), y(0))$, i.e.,

$$V(e^{\alpha(k-1)}, \alpha) < V(k, \alpha k).$$

However, this inequality cannot hold for k large enough, since for a generic fixed α, the left hand side grows exponentially in k whereas the right hand side grows only polynomially in k. The only subtlety arises from the fact that $V(e^{\alpha(k-1)}, \alpha)$ could potentially be a constant for some particular choices of α. However, for any polynomial $V(x, y)$ with nontrivial dependence on y, this may happen for at most finitely many values of α. Therefore, any generic choice of α would make the argument work. ∎

II. Conclusions

We showed that existence of a polynomial Lyapunov function is not necessary for global asymptotic stability of a polynomial vector field. It would be interesting to determine whether *globally exponentially stable* (GES) systems admit polynomial Lyapunov functions. Our counterexample is certainly not GES. Another related converse Lyapunov question that is motivated by the use of computational techniques for analysis via polynomial Lyapunov functions is the following:

Question: Suppose a polynomial vector field has a polynomial Lyapunov function. Does this imply that sum of squares optimization will succeed in finding a polynomial Lyapunov function and proving stability?

Some results on this question are presented in recent work [7] and [8].

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Appendix: Example of Bacciotti and Rosier

After our counterexample was submitted for publication, Christian Ebenbauer brought to our attention an earlier counterexample that appears in a book by Bacciotti and Rosier [9, Prop. 5.2] and achieves the same goal (though by using irrational coefficients). We explain the differences between the two examples below.

The example in [9] is a vector field in 2 variables and degree 5 that is GAS but has no polynomial (and no analytic) Lyapunov function even around the origin. The construction there is complementary to ours: while the cause of the lack of existence of polynomial Lyapunov functions in our Theorem 1.1 is fast growth *arbitrarily far* away from the origin, the obstacle in their example is slow decay *arbitrarily close* to the origin. As one would expect, the vector field given in [9] is not locally exponentially stable. By contrast, the linearization of the vector field in (1) is Hurwitz, so our system is locally exponentially stable.

The example in [9] crucially relies on *irrationality* of a parameter that appears as part of the coefficients of the vector field. (Indeed, one easily observes that if that parameter is rational, then the vector field does admit a polynomial Lyapunov function.) In practical applications where computational techniques for searching over Lyapunov functions on finite precision machines are used, such issues with irrationality of the input cannot occur. By contrast, the example in (1) demonstrates that non-existence of polynomial Lyapunov functions can happen for extremely simple systems that may very well appear in applications.

As we remarked earlier, it is known that locally exponentially stable polynomial vector fields admit polynomial Lyapunov functions on compact sets [3]. The example in [9] implies that the assumption of exponential stability indeed cannot be dropped.